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# A Certain Subclass of Analytic Functions Involving Operators of Fractional Calculus

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**Abstract**—Making use of certain operators of fractional calculus, we introduce a new class $\mathcal{T}_\mu(n,\lambda,\alpha)$ of functions which are analytic and univalent in the open unit disk $\mathcal{U}$ and obtain a necessary and sufficient condition for a function to be in the class $\mathcal{T}_\mu(n,\lambda,\alpha)$. The various other results presented here for the class $\mathcal{T}_\mu(n,\lambda,\alpha)$ include the radii of close-to-convexity, starlikeness, and convexity, and some growth and distortion theorems involving fractional integrals and fractional derivatives. Some interesting consequences and possible further generalizations of these results are also pointed out.

**Keywords**—Analytic functions, Univalent functions, Fractional calculus (fractional integral and fractional derivative), Close-to-convex functions, Starlike functions, Convex functions, Maximum modulus theorem, Growth and distortion theorems, Gauss hypergeometric function.

## 1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{T}(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \qquad (a_k \geq 0;\ n \in \mathbb{N} := \{1,2,3,\dots\}), \tag{1.1}$$

which are *analytic* and *univalent* in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $\mathcal{T}_\mu(n,\lambda,\alpha)$ the subclass of functions $f(z)$ in $\mathcal{T}(n)$ which also satisfy the inequality

$$\left| \frac{z\, D_z^{1+\mu} f(z) - (1-\mu)\, D_z^{\mu} f(z)}{\lambda z\, D_z^{1+\mu} f(z) + (1-\lambda)\, D_z^{\mu} f(z)} \right| < \alpha, \qquad (z \in \mathcal{U};\ 0 < \alpha \leq 1;\ 0 \leq \lambda \leq 1;\ 0 \leq \mu < 1), \tag{1.2}$$

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where, and in what follows, $D_z^\mu$ denotes an operator of fractional calculus (that is, fractional integral and fractional derivative), given by Definitions 1 and 2 below (cf., e.g., [1–3]).

DEFINITION 1. (FRACTIONAL INTEGRAL OPERATOR). *The fractional integral of order $\mu$ is defined, for a function $f(z)$, by*

$$D_z^{-\mu} f(z) := \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}}\, d\zeta, \qquad (\mu > 0), \tag{1.3}$$

*where $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.*

DEFINITION 2. (FRACTIONAL DERIVATIVE OPERATOR). *The fractional derivative of order $\mu$ is defined, for a function $f(z)$, by*

$$D_z^\mu f(z) := \begin{cases} \dfrac{1}{\Gamma(1-\mu)} \dfrac{d}{dz} \int_0^z \dfrac{f(\zeta)}{(z-\zeta)^\mu}\, d\zeta, & (0 \le \mu < 1), \\ \dfrac{d^m}{dz^m} D_z^{\mu-m} f(z), & (m \le \mu < m+1;\ m \in \mathbb{N}), \end{cases} \tag{1.4}$$

*where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^\mu$ is removed, as in Definition 1.*

In our systematic investigation of the various properties and characteristics of the class $\mathcal{T}_\mu(n, \lambda, \alpha)$, we shall also require the use of a number of other classes of functions associated with $\mathcal{T}(n)$. First of all, a function $f(z) \in \mathcal{T}(n)$ is said to be *close-to-convex of order* $\alpha$ in $\mathcal{U}$ if it satisfies the inequality

$$\Re\{f'(z)\} > \alpha, \qquad (z \in \mathcal{U};\ 0 \le \alpha < 1). \tag{1.5}$$

On the other hand, a function $f(z) \in \mathcal{T}(n)$ is said to be *starlike of order* $\alpha$ in $\mathcal{U}$ if it satisfies the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in \mathcal{U};\ 0 \le \alpha < 1). \tag{1.6}$$

Furthermore, a function $f(z) \in \mathcal{T}(n)$ is said to be *convex of order* $\alpha$ in $\mathcal{U}$ if and only if $zf'(z)$ is starlike of order $\alpha$, that is, if it satisfies the inequality

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (z \in \mathcal{U};\ 0 \le \alpha < 1). \tag{1.7}$$

(See, for details, [3,4].)

In view of the above definitions, it is easily verified that the class $\mathcal{T}_\mu(n, \alpha, \lambda)$ can be identified with the class of

(i) starlike functions of order $1-\alpha$ when

$$\lambda = \mu = 0; \tag{1.8}$$

(ii) convex functions of order $1-\alpha$ when

$$\lambda = 0 \quad \text{and} \quad \mu \to 1-. \tag{1.9}$$

For other subclasses of analytic functions $f(z)$ of the form (1.1), which are also defined by using operators of fractional calculus, one may refer to the *recent* works of Altintaş *et al.* [5], Raina and Srivastava [6], and others.

## 2. A CHARACTERIZATION OF THE CLASS $\mathcal{T}_\mu(\mathrm{n},\alpha,\lambda)$

We begin by proving a characterization property of the class $\mathcal{T}_\mu(n,\alpha,\lambda)$, which is contained in the following theorem.

THEOREM 1. *Let the function $f(z) \in \mathcal{T}(n)$ be defined by (1.1). Then, $f(z)$ is in the class $\mathcal{T}_\mu(n,\alpha,\lambda)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{\{k+\alpha\,[\lambda(k-\mu-1)+1]-1\}\,\Gamma(k+1)}{\Gamma(k-\mu+1)}\,a_k \le \frac{\alpha(1-\lambda\mu)}{\Gamma(2-\mu)}, \tag{2.1}$$
$$(n \in \mathbb{N};\ 0<\alpha\le 1;\ 0\le\lambda\le 1;\ 0\le\mu<1).$$

*The result is sharp, the extremal function being given by*

$$f(z) = z - \frac{\alpha(1-\lambda\mu)\,\Gamma(n-\mu+2)}{\{n+\alpha\,[\lambda(n-\mu)+1]\}\,\Gamma(n+2)\,\Gamma(2-\mu)}\,z^{n+1}, \qquad (n\in\mathbb{N}). \tag{2.2}$$

PROOF. First of all, it is easily seen from Definition 2 that

$$D_z^{\mu}\{z^{\kappa}\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\mu+1)}\,z^{\kappa-\mu}, \qquad (0\le\mu<1;\ \kappa>-1). \tag{2.3}$$

Now, suppose that the function $f(z) \in \mathcal{T}(n)$ is defined by (1.1) and that the inequality (2.1) holds true. Then making use of the fractional derivative formula (2.3), we find for $z \in \partial\mathcal{U}$ that

$$\begin{aligned}
&\left|z^{\mu}\,D_z^{1+\mu}f(z) - (1-\mu)\,z^{\mu-1}\,D_z^{\mu}f(z)\right| - \alpha\left|\lambda z^{\mu}\,D_z^{1+\mu}f(z) + (1-\lambda)\,z^{\mu-1}\,D_z^{\mu}f(z)\right| \\
&= \left|-\sum_{k=n+1}^{\infty}\frac{(k-1)\,\Gamma(k+1)}{\Gamma(k-\mu+1)}\,a_k\,z^{k-1}\right| - \alpha\left|\frac{1-\lambda\mu}{\Gamma(2-\mu)} - \sum_{k=n+1}^{\infty}\frac{[\lambda(k-\mu-1)+1]\,\Gamma(k+1)}{\Gamma(k-\mu+1)}\,a_k\,z^{k-1}\right| \\
&\le \sum_{k=n+1}^{\infty}\frac{\{k+\alpha\,[\lambda(k-\mu-1)+1]-1\}\,\Gamma(k+1)}{\Gamma(k-\mu+1)}\,a_k - \frac{\alpha(1-\lambda\mu)}{\Gamma(2-\mu)} \le 0,
\end{aligned} \tag{2.4}$$

by virtue of the inequality (2.1). Hence by the maximum modulus theorem, $f(z)$ belongs to the class $\mathcal{T}_\mu(n,\alpha,\lambda)$.

To prove the converse, we assume that $f(z)$ is defined by (1.1) and is in the class $\mathcal{T}_\mu(n,\alpha,\lambda)$, so that the condition (1.2) readily yields

$$\begin{aligned}
&\left|\frac{z\,D_z^{1+\mu}f(z) - (1-\mu)\,D_z^{\mu}f(z)}{\lambda z\,D_z^{1+\mu}f(z) + (1-\lambda)\,D_z^{\mu}f(z)}\right| \\
&= \left|\frac{-\displaystyle\sum_{k=n+1}^{\infty}\frac{(k-1)\,\Gamma(k+1)\,\Gamma(2-\mu)}{\Gamma(k-\mu+1)}\,a_k\,z^{k-1}}{(1-\lambda\mu) - \displaystyle\sum_{k=n+1}^{\infty}\frac{[\lambda(k-\mu-1)+1]\,\Gamma(k+1)\,\Gamma(2-\mu)}{\Gamma(k-\mu+1)}\,a_k\,z^{k-1}}\right| < \alpha
\end{aligned} \tag{2.5}$$
$$(z\in\mathcal{U};\ 0<\alpha\le 1;\ 0\le\lambda\le 1;\ 0\le\mu<1;\ n\in\mathbb{N}).$$

Since $|\Re(z)| \le |z|$ for any $z$, if we choose $z$ to be real and let $z \to 1-$, we shall find from (2.5) that

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\frac{(k-1)\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k-\mu+1)}\,a_k \\
&\qquad \le \alpha\left\{(1-\lambda\mu) - \sum_{k=n+1}^{\infty}\frac{[\lambda(k-\mu-1)+1]\,\Gamma(k+1)\,\Gamma(2-\mu)}{\Gamma(k-\mu+1)}\,a_k\right\},
\end{aligned} \tag{2.6}$$

which readily yields the desired assertion (2.1).

Finally by observing that the function $f(z)$ defined by (2.2) is indeed an extremal function for the assertion (2.1), we complete the proof of Theorem 1.

An immediate consequence of Theorem 1 may be stated as the following corollary.

COROLLARY 1. *If the function $f(z)$ defined by (1.1) is in the class $\mathcal{T}_\mu(n,\alpha,\lambda)$, then*

$$a_k \le \frac{\alpha(1-\lambda\mu)\,\Gamma(k-\mu+1)}{\{k+\alpha\,[\lambda(k-\mu-1)+1]-1\}\;\Gamma(k+1)\,\Gamma(2-\mu)}, \qquad (k=n+1,n+2,n+3,\ldots;\; n\in\mathbb{N}). \tag{2.7}$$

Next, we prove the following theorem.

THEOREM 2. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by*

$$g(z) = z - \sum_{k=n+1}^{\infty} b_k\, z^k, \qquad (b_k \ge 0;\; n\in\mathbb{N}) \tag{2.8}$$

*be in the same class $\mathcal{T}_\mu(n,\alpha,\lambda)$. Then, the function $h(z)$ defined by*

$$h(z) := (1-\xi)\,f(z) + \xi\, g(z) = z - \sum_{k=n+1}^{\infty} c_k\, z^k, \qquad (c_k := (1-\xi)\,a_k + \xi\, b_k \ge 0;\; 0\le\xi\le 1;\; n\in\mathbb{N}) \tag{2.9}$$

*is also in the class $\mathcal{T}_\mu(n,\alpha,\lambda)$.*

PROOF. The assertion of Theorem 2 would follow easily by making use of (2.1), (2.8), and (2.9).

More generally, we can similarly prove the following theorem.

THEOREM 3. *Let each of the functions $f_j(z)$ defined by*

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{k,j}\, z^k, \qquad (a_{k,j}\ge 0;\; j=1,\ldots,m;\; n\in\mathbb{N}) \tag{2.10}$$

*be in the same class $\mathcal{T}_\mu(n,\alpha,\lambda)$. Then, the function $h(z)$ defined by*

$$h(z) := \sum_{j=1}^{m} c_j\, f_j(z), \tag{2.11}$$

*where*

$$c_j \ge 0, \quad (j=1,\cdots,m) \qquad \text{and} \qquad \sum_{j=1}^{m} c_j = 1, \tag{2.12}$$

*is also in the class $\mathcal{T}_\mu(n,\alpha,\lambda)$.*

## 3. GROWTH AND DISTORTION THEOREMS

In this section, we prove some growth and distortion theorems associated with the operators of fractional calculus, which were introduced in Section 1. We first state the following theorem.

THEOREM 4. *If $f(z)\in\mathcal{T}_\mu(n,\alpha,\lambda)$, then*

$$\begin{aligned}\left(1-\frac{\alpha(1-\lambda\mu)\Gamma(n-\mu+2)\Gamma(\nu+2)}{\{n+\alpha[\lambda(n-\mu)+1]\}\Gamma(2-\mu)\Gamma(\nu+n+2)}\,|z|^n\right)\frac{|z|^{\nu+1}}{\Gamma(\nu+2)} &\le \left|D_z^{-\nu}f(z)\right| \\ \le \left(1+\frac{\alpha(1-\lambda\mu)\Gamma(n-\mu+2)\Gamma(\nu+2)}{\{n+\alpha[\lambda(n-\mu)+1]\}\Gamma(2-\mu)\Gamma(\nu+n+2)}\,|z|^n\right)&\frac{|z|^{\nu+1}}{\Gamma(\nu+2)},\end{aligned} \tag{3.1}$$

$$(z \in \mathcal{U};\ 0 < \alpha \leq 1;\ 0 \leq \lambda \leq 1;\ 0 \leq \mu < 1;\ l, \nu > 0;\ n \in \mathbb{N}). \tag{3.1(cont.)}$$

*The result is sharp for the function $f(z)$ given by (2.2).*

PROOF. Since $f(z) \in \mathcal{T}_\mu(n, \alpha, \lambda)$, by applying the assertion (2.1) of Theorem 1, we obtain

$$\begin{aligned}\frac{n + \alpha[\lambda(n-\mu)+1]}{\Gamma(n-\mu+2)} \sum_{k=n+1}^{\infty} \Gamma(k+1)\, a_k &\leq \sum_{k=n+1}^{\infty} \frac{\{k + \alpha[\lambda(k-\mu-1)+1] - 1\}\,\Gamma(k+1)}{\Gamma(k-\mu+1)}\, a_k \\ &\leq \frac{\alpha(1-\lambda\mu)}{\Gamma(2-\mu)},\end{aligned}$$

which immediately yields

$$\sum_{k=n+1}^{\infty} \Gamma(k+1)\, a_k \leq \frac{\alpha(1-\lambda\mu)\,\Gamma(n-\mu+2)}{\{n + \alpha[\lambda(n-\mu)+1]\}\,\Gamma(2-\mu)}, \qquad (n \in \mathbb{N}). \tag{3.2}$$

Making use of Definition 1, we get

$$D_z^{-\nu} f(z) = \frac{z^{\nu+1}}{\Gamma(\nu+2)} \left(1 - \sum_{k=n+1}^{\infty} \Theta(k)\,\Gamma(k+1)\, a_k\, z^{k-1}\right), \tag{3.3}$$

where for convenience,

$$\Theta(k) := \frac{\Gamma(\nu+2)}{\Gamma(k+\nu+1)}, \qquad (k = n+1, n+2, n+3, \dots;\ n \in \mathbb{N};\ \nu > 0).$$

Clearly the function $\Theta(k)$ is decreasing in $k$, and we have

$$0 < \Theta(k) \leq \Theta(n+1) = \frac{\Gamma(\nu+2)}{\Gamma(\nu+n+2)}, \qquad (k = n+1, n+2, n+3, \dots;\ n \in \mathbb{N};\ \nu > 0). \tag{3.4}$$

Thus from (3.2)–(3.4), it is easily seen that

$$\begin{aligned}\left|D_z^{-\nu} f(z)\right| &\leq \frac{|z|^{\nu+1}}{\Gamma(\nu+2)} \left(1 + |z|^n\, \Theta(n+1) \sum_{k=n+1}^{\infty} \Gamma(k+1)\, a_k\right) \\ &\leq \frac{|z|^{\nu+1}}{\Gamma(\nu+2)} \left(1 + \frac{\alpha(1-\lambda\mu)\,\Gamma(n-\mu+2)\,\Gamma(\nu+2)}{\{n + \alpha[\lambda(n-\mu)+1]\}\,\Gamma(2-\mu)\,\Gamma(\nu+n+2)}\, |z|^n\right),\end{aligned}$$

and that

$$\begin{aligned}\left|D_z^{-\nu} f(z)\right| &\geq \frac{|z|^{\nu+1}}{\Gamma(\nu+2)} \left(1 - |z|^n\, \Theta(n+1) \sum_{k=n+1}^{\infty} \Gamma(k+1)\, a_k\right) \\ &\geq \frac{|z|^{\nu+1}}{\Gamma(\nu+2)} \left(1 - \frac{\alpha(1-\lambda\mu)\,\Gamma(n-\mu+2)\,\Gamma(\nu+2)}{\{n + \alpha[\lambda(n-\mu)+1]\}\,\Gamma(2-\mu)\,\Gamma(\nu+n+2)}\, |z|^n\right),\end{aligned}$$

which evidently complete the proof of Theorem 4.

By making use of Definition 2 (instead of Definition 1), we can similarly prove Theorem 5 and Theorem 6 below.

THEOREM 5. *If $f(z) \in \mathcal{T}_\mu(n, \alpha, \lambda)$, then*

$$\begin{aligned}\frac{|z|^{1-\nu}}{\Gamma(2-\nu)} &\left(1 - \frac{\alpha(1-\lambda\mu)\,\Gamma(n-\mu+2)\,\Gamma(2-\nu)}{\{n + \alpha[\lambda(n-\mu)+1]\}\,\Gamma(2-\mu)\,\Gamma(n-\nu+2)}\, |z|^n\right) \leq |D_z^{\nu} f(z)| \\ &\leq \frac{|z|^{1-\nu}}{\Gamma(2-\nu)} \left(1 + \frac{\alpha(1-\lambda\mu)\,\Gamma(n-\mu+2)\,\Gamma(2-\nu)}{\{n + \alpha[\lambda(n-\mu)+1]\}\,\Gamma(2-\mu)\,\Gamma(n-\nu+2)}\, |z|^n\right),\end{aligned} \tag{3.5}$$

$$(z \in \mathcal{U};\ 0 < \alpha \leq 1;\ 0 \leq \lambda \leq 1;\ 0 \leq \mu < 1;\ 0 \leq \nu < 1;\ n \in \mathbb{N}).$$

THEOREM 6. *If $f(z) \in \mathcal{T}_\mu(n,\alpha,\lambda)$, then*

$$\begin{aligned} \frac{|z|^{-\nu}}{\Gamma(1-\nu)}\left(1-\frac{\alpha(1-\lambda\mu)\,\Gamma(n-\mu+2)\,\Gamma(1-\nu)}{\{n+\alpha[\lambda(n-\mu)+1]\}\,\Gamma(2-\mu)\,\Gamma(n-\nu+1)}|z|^n\right) &\le \left|D_z^{1+\nu}f(z)\right| \\ \le \frac{|z|^{-\nu}}{\Gamma(1-\nu)}\left(1+\frac{\alpha(1-\lambda\mu)\,\Gamma(n-\mu+2)\,\Gamma(1-\nu)}{\{n+\alpha[\lambda(n-\mu)+1]\}\,\Gamma(2-\mu)\,\Gamma(n-\nu+1)}|z|^n\right), \end{aligned} \tag{3.6}$$

$$(z\in\mathcal{U}\setminus\{0\};\ 0<\alpha\le 1;\ 0\le\lambda\le 1;\ 0\le\mu<1;\ 0\le\nu<1;\ n\in\mathbb{N}),$$

*it being understood that $z\in\mathcal{U}$ for $\nu=0$.*

The result is sharp for the function $f(z)$ given by (2.2).

## 4. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS, AND CONVEXITY

THEOREM 7. *If $f(z)\in\mathcal{T}_\mu(n,\alpha,\lambda)$, then $f(z)$ is close-to-convex of order $\beta$ $(0\le\beta<1)$ in $|z|<r_1$, where*

$$\begin{aligned} r_1 &= r_1(\alpha,\beta,\lambda,\mu) \\ &:= \inf_k\left[\frac{(1-\beta)\{k+\alpha[\lambda(k-\mu-1)+1]-1\}\Gamma(k)\Gamma(2-\mu)}{\alpha(1-\lambda\mu)\,\Gamma(k-\mu+1)}\right]^{1/(k-1)}, \\ &\qquad (k=n+1,n+2,n+3,\dots;\ n\in\mathbb{N}). \end{aligned} \tag{4.1}$$

*The result is sharp for the functions $f(z)$ given by*

$$f(z)=z-\frac{\alpha(1-\lambda\mu)\Gamma(k-\mu+1)}{\{k+\alpha[\lambda(k-\mu-1)+1]-1\}\Gamma(k+1)\Gamma(2-\mu)},\qquad (k\ge n+1;\ n\in\mathbb{N}). \tag{4.2}$$

PROOF. Let $f(z)\in\mathcal{T}_\mu(n,\alpha,\lambda)$. Then by virtue of (1.5), the function $f(z)$ defined by (1.1) is close-to-convex of order $\beta$ in $|z|<r_1$, provided that

$$\begin{aligned} |f'(z)-1| &= \left|\sum_{k=n+1}^{\infty} k\,a_k\,z^{k-1}\right| \\ &\le \sum_{k=n+1}^{\infty} k\,a_k\,|z|^{k-1} \\ &\le 1-\beta, \qquad (|z|<r_1;\ 0\le\beta<1), \end{aligned} \tag{4.3}$$

where $r_1$ is given by (4.1).

But in view of (2.1), this last inequality (4.3) holds true if

$$\frac{k\,|z|^{k-1}}{1-\beta}\le\frac{\{k+\alpha[\lambda(k-\mu-1)+1]-1\}\,\Gamma(k+1)\,\Gamma(2-\mu)}{\alpha(1-\lambda\mu)\,\Gamma(k-\mu+1)},$$
$$(k=n+1,n+2,n+3,\dots;\ n\in\mathbb{N}),$$

which, when solved for $|z|$, yields (4.1).

THEOREM 8. *If $f(z)\in\mathcal{T}_\mu(n,\alpha,\lambda)$, then $f(z)$ is starlike of order $\beta$ $(0\le\beta<1)$ in $|z|<r_2$, where*

$$\begin{aligned} r_2 &= r_2(\alpha,\beta,\lambda,\mu) \\ &:= \inf_k\left[\frac{(1-\beta)\,\{k+\alpha\,[\lambda(k-\mu-1)+1]-1\}\,\Gamma(k+1)\,\Gamma(2-\mu)}{\alpha(1-\lambda\mu)(k-\beta)\Gamma(k-\mu+1)}\right]^{1/(k-1)}, \\ &\qquad (k=n+1,n+2,n+3,\dots;\ n\in\mathbb{N}). \end{aligned} \tag{4.4}$$

*The result is sharp for the functions $f(z)$ given by (4.2).*

PROOF. Under the hypothesis of Theorem 8, we must show that

$$\left|\frac{z\,f'(z)}{f(z)}-1\right| \leq 1-\beta, \qquad (|z|<r_2;\ 0\leq\beta<1),$$

where $r_2$ is given by (4.4). In fact, we have

$$\left|\frac{z\,f'(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{k=n+1}^{\infty}(k-1)\,a_k\,|z|^{k-1}}{1-\sum\limits_{k=n+1}^{\infty}a_k\,|z|^{k-1}} \leq 1-\beta,$$

provided that

$$\frac{(k-\beta)\,|z|^{k-1}}{1-\beta} \leq \frac{\{k+\alpha\,[\lambda(k-\mu-1)+1]-1\}\,\Gamma(k+1)\,\Gamma(2-\mu)}{\alpha(1-\lambda\mu)\,\Gamma(k-\mu+1)}, \qquad (k=n+1,n+2,n+3,\ldots;\ n\in\mathbb{N}), \tag{4.5}$$

where we have also applied the inequality (2.1).

Upon solving (4.5) for $|z|$, we readily obtain (4.4).

THEOREM 9. *If $f(z)\in\mathcal{T}_\mu(n,\alpha,\lambda)$, then $f(z)$ is convex of order $\beta$ in $|z|<r_3$, where*

$$\begin{aligned} r_3 &= r_3(\alpha,\beta,\lambda,\mu) \\ &:= \inf_k \left[\frac{(1-\beta)\,\{k+\alpha\,[\lambda(k-\mu-1)+1]-1\}\,\Gamma(k)\,\Gamma(2-\mu)}{\alpha(1-\lambda\mu)\,(k-\beta)\,\Gamma(k-\mu+1)}\right]^{1/(k-1)} \\ &\qquad (k=n+1,n+2,n+3,\ldots;\ n\in\mathbb{N}). \end{aligned} \tag{4.6}$$

*The result is sharp for the functions $f(z)$ given by (4.2).*

PROOF. Under the hypothesis of Theorem 9, it is sufficient to show that

$$\left|\frac{z\,f''(z)}{f'(z)}\right| \leq 1-\beta, \qquad (|z|<r_3;\ 0\leq\beta<1), \tag{4.7}$$

where $r_3$ is given by (4.6). Since

$$\left|\frac{z\,f''(z)}{f'(z)}\right| \leq \frac{\sum_{k=n+1}^{\infty}k(k-1)\,a_k\,|z|^{k-1}}{1-\sum_{k=n+1}^{\infty}k\,a_k\,|z|^{k-1}}, \tag{4.8}$$

the inequality (4.7) does hold true in

$$\frac{k(k-\beta)\,|z|^{k-1}}{1-\beta} \leq \frac{\{k+\alpha\,[\lambda(k-\mu-1)+1]-1\}\,\Gamma(k+1)\,\Gamma(2-\mu)}{\alpha(1-\lambda\mu)\,\Gamma(k-\mu+1)}, \qquad (k=n+1,n+2,n+3,\cdots;\ n\in\mathbb{N}), \tag{4.9}$$

by virtue of the assertion (2.1) of Theorem 1, and we are led easily to (4.6).

## 5. REMARKS AND OBSERVATIONS

By assigning suitable particular values to the parameters $\lambda$ and $\mu$, especially as indicated in (1.8) and (1.9), we can derive from each of our theorems the corresponding results for several simpler subclasses of the class $\mathcal{T}(n)$. Furthermore, the parameter $\nu$ occurring in Theorems 4–6, and the parameter $\beta$ occurring in Theorems 7–9, can be suitably specialized to obtain a number of particular cases of these results. For example, by setting $\nu=0$ in Theorem 6 (or, alternatively, by letting $\nu \to 1-$ in Theorem 5), we readily obtain the following distortion theorem for the class $\mathcal{T}_\mu(n,\alpha,\lambda)$.

COROLLARY 2. *If $f(z) \in \mathcal{T}_\mu(n,\alpha,\lambda)$, then*

$$\begin{aligned} 1 - \frac{\alpha(1-\lambda\mu)\,\Gamma(n-\mu+2)}{\{n+\alpha\,[\lambda(n-\mu)+1]\}\,\Gamma(2-\mu)\,\Gamma(n+1)}\,|z|^n &\leq |f'(z)| \\ &\leq 1 + \frac{\alpha(1-\lambda\mu)\,\Gamma(n-\mu+2)}{\{n+\alpha\,[\lambda(n-\mu)+1]\}\,\Gamma(2-\mu)\,\Gamma(n+1)}\,|z|^n, \end{aligned} \tag{5.1}$$

$$(z \in \mathcal{U};\ 0 < \alpha \leq 1;\ 0 \leq \lambda \leq 1;\ 0 \leq \mu < 1;\ n \in \mathbb{N}).$$

*The result is sharp for the function $f(z)$ given by (2.2).*

In terms of the generalized fractional calculus operators $I_{0,z}^{\mu,\rho,\sigma}$ and $J_{0,z}^{\mu,\rho,\sigma}$, which are defined with the Gauss hypergeometric function in the kernels (cf., e.g., [6] as well as the references cited there), the class $\mathcal{T}_\mu(n,\alpha,\lambda)$ can indeed be generalized further by using the fractional derivative operator $J_{0,z}^{\mu,\rho,\sigma}$ in the condition (1.3) in place of $D_z^\mu$ (and Theorems 4–6 can be extended to yield growth and distortion theorems involving the operators $I_{0,z}^{\nu,\gamma,\delta}$ and $J_{0,z}^{\nu,\gamma,\delta}$ appropriately). These generalizations of the work presented here are fairly straightforward. For the sake of ready reference, we simply state the definitions of the generalized fractional calculus operators $I_{0,z}^{\mu,\rho,\sigma}$ and $J_{0,z}^{\mu,\rho,\sigma}$.

DEFINITION 3. *Under the hypotheses of Definition 1, the generalized fractional integral of order $\mu$ is defined, for a function $f(z)$, by*

$$\begin{gathered} I_{0,z}^{\mu,\rho,\sigma} f(z) := \frac{z^{-\mu-\rho}}{\Gamma(\mu)} \int_0^z (z-\zeta)^{\mu-1}\, {}_2F_1\left(\mu+\rho, -\sigma; \mu; 1-\frac{\zeta}{z}\right) f(\zeta)\,d\zeta, \\ (\mu > 0;\ \eta > \max\{0, \rho-\sigma\} - 1) \end{gathered} \tag{5.2}$$

*and the generalized fractional derivative of order $\mu$ is defined, for a function $f(z)$, by*

$$J_{0,z}^{\mu,\rho,\sigma} f(z) := \begin{cases} \dfrac{1}{\Gamma(1-\mu)} \dfrac{d}{dz}\Big\{ z^{\mu-\rho} \int_0^z (z-\zeta)^{-\mu} \\ \qquad \cdot {}_2F_1\left(\rho-\mu, 1-\sigma; 1-\mu; 1-\frac{\zeta}{z}\right) f(\zeta)\,d\zeta \Big\}, & (0 \leq \mu < 1), \\ \dfrac{d^m}{dz^m} J_{0,z}^{\mu-m,\rho,\sigma} f(z), & (m \leq \mu < m+1;\ m \in \mathbb{N}), \end{cases} \tag{5.3}$$
$$(\eta > \max\{0,\ \rho-\sigma\} - 1),$$

*provided further that*

$$f(z) = O\left(|z|^\eta\right), \qquad (z \to 0). \tag{5.4}$$

By comparing Definition 1 with (5.2), we obtain the relationship:

$$D_z^{-\mu} f(z) = I_{0,z}^{\mu,-\mu,\sigma} f(z), \qquad (\mu > 0). \tag{5.5}$$

Similarly by comparing Definition 2 with (5.3), we find that

$$D_z^{\mu} f(z) = J_{0,z}^{\mu,\mu,\sigma} f(z), \qquad (0 \leq \mu < 1). \tag{5.6}$$

It is such relationships as (5.5) and (5.6) that would enable one to generalize each of the results presented here by simply introducing the fractional calculus operators $I_{0,z}^{\mu,\rho,\sigma}$ and $J_{0,z}^{\mu,\rho,\sigma}$ appropriately.

# REFERENCES


1. S. Owa, On the distortion theorems. I, *Kyungpook Math. J.* **18**, 53–59 (1978).
2. S. Owa and H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* **39**, 1057–1077 (1987).
3. H.M. Srivastava and S. Owa, Editors, *Current Topics in Analytic Function Theory*, World Scientific, Singapore, (1992).
4. P.L. Duren, Univalent functions, *Grundlehren der Mathematischen Wissenschaften*, Bd. 259, Springer-Verlag, New York, (1983).
5. O. Altintaş, H. Irmak and H.M. Srivastava, A subclass of analytic functions defined by using certain operators of fractional calculus, *Computers Math. Applic.* **30** (1), 1–9 (1995).
6. R.K. Raina and H.M. Srivastava, A certain subclass of analytic functions associated with operators of fractional calculus, *Computers Math. Applic.* **32** (7), 13–19 (1996).